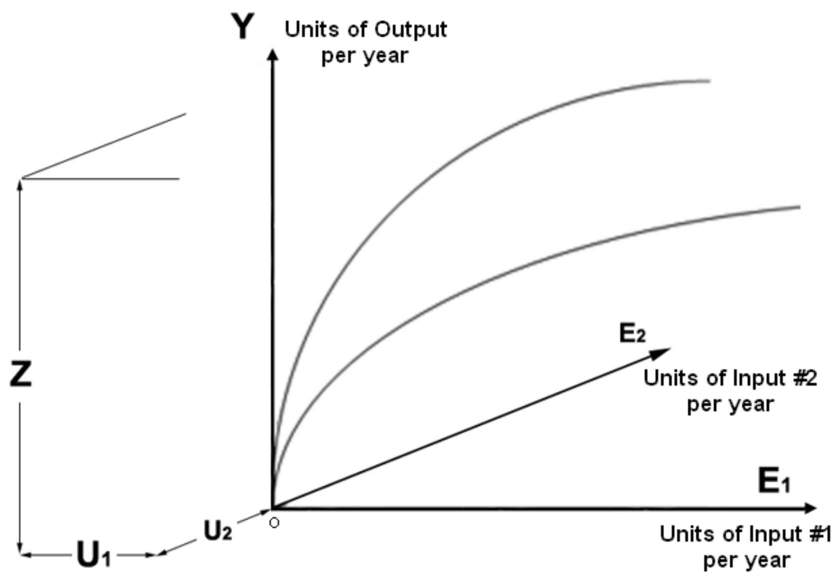
Hyperbolic Descriptions of Technical Indifference
and the
Polynomial Factoring Problem

SFEcon is unique in its use of hyperbolic systems to describe the relations among an economic sector's output rates Y and the input rates E_J of its productive factors J. As shown in the figure below, loci of technical optima, i.e. the set [Y,E_J], are shaped by a set of utility parameters [Z,U_J] so as to express diminishing marginal utility. The set [Z,U_J] relates a production or utility function's origin to the point at which all asymptotes to the hyperbolic form intersect.



For this parametric system, an economic sector's physical output rate Y would relate to its N physical input rates E_J by the following equation:

1) $$Y = Z \cdot \left[1 - \frac{U_1 \cdot U_2 \cdot \ldots \cdot U_N}{(U_1 + E_1) \cdot (U_2 + E_2) \cdot \ldots \cdot (U_N + E_N)} \right]$$

And the hyperbolic system's marginal products would be disclosed thusly:

2) $$\frac{\partial Y}{\partial E_J} = \frac{Z - Y}{U_J + E_J}$$

Hyperbolic forms, because of their unique properties in the derivative, suggest themselves wherever a natural process of dynamic accumulation and development requires description. These functions can be shown to provide general and mathematically closed-form solutions to economics' most primitive and defining problem, i.e.: isolating an economic optimum from among technical optima.

Solution to this problem requires knowledge of the price environment [π,P_J] where π is the price of the good being produced, and P_J represents a vector of prices for the factor inputs J. Optimal criteria require that each input's value of marginal product $\pi \cdot dY/dE_J$ equal the price P_J of the input commodity. Stated in terms of the hyperbolic system's equation for marginal product, this means …

3) $$\pi \cdot \frac{Z - Y}{U_J + E_J} = P_J$$

for all J = 1 to N. The optimal relationship between [Y,E_J] and [Z,U_J] would be governed by N such equations, where N is the number of inputs. A simultaneous system relating [Y,E_J] with [Z,U_J] at [π,P_J] is completed by requiring that the optimal [Y,E_J] cooperate with [Z,U_J] in an exact solution to the production function of Equation 1.

General solutions to such systems have always been thought impossible given the degrees of curvature presented by the production function. The system requires simultaneous solution of N+1 equations in N+1 unknowns, where each equation would be of degree N+1. As this would create insurmountable 'polynomial factoring problems' for any N >3, possibilities for practical solutions have remained beneath consideration.

The hyperbolic system does, however, offer a general and exact solution to the specific problem of economic optimality. It unfolds by identifying ζ, the central equality in Equation 3's relation of marginal value to price:

4) $$\begin{aligned}\zeta &= \pi \cdot (Z - Y) \\ &= P_1 \cdot (U_1 + E_1) \\ &= P_2 \cdot (U_2 + E_2) \\ &\vdots \\ &= P_N \cdot (U_N + E_N)\end{aligned}$$

Solution proceeds by rearranging Equation 1's production function:

5) $$1 = \frac{Z}{(Z - Y)} \cdot \frac{U_1}{(U_1 + E_1)} \cdot \frac{U_2}{(U_2 + E_2)} \cdot \ldots \cdot \frac{U_N}{(U_N + E_N)}$$

Corresponding unit ratios [π/π, P_J/P_J] are then applied to each term in Equation 5:

6) $$1 = \frac{\pi \cdot Z}{\pi \cdot (Z - Y)} \cdot \frac{P_1 \cdot U_1}{P_1 \cdot (U_1 + E_1)} \cdot \frac{P_2 \cdot U_2}{P_2 \cdot (U_2 + E_2)} \cdot \ldots \cdot \frac{P_N \cdot U_N}{P_N \cdot (U_N + E_N)}$$

Equation 4 then provides substitutions of ζ for each denominator in the above:

7) $$1 = \frac{\pi \cdot Z}{\zeta} \cdot \frac{P_1 \cdot U_1}{\zeta} \cdot \frac{P_2 \cdot U_2}{\zeta} \cdot \ldots \cdot \frac{P_N \cdot U_N}{\zeta}$$

Solving for ζ then yields:

8) $$\zeta = (\pi \cdot Z \cdot P_1 \cdot U_1 \cdot P_2 \cdot U_2 \cdot \ldots \cdot P_N \cdot U_N)^{\frac{1}{N+1}}$$

where the classic polynomial factoring problem is reduced to extraction of a higher-ordered root. If an economic sector's price environment [π,P_J] and utility tradeoffs [Z,U_J] are known, then the optimal values of [Y,E_J] are disclosed in mathematically closed-form: once ζ is calculated, all optimal elements of [Y,E_J] are determined by Equation 4.